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Graphs Colouring

Book of abstracts

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List of talks

Gauss words and rhythmic canons

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A *rhythmic canon* is a combinatorial structure consisting of repeating copies of the same *motif*. These copies may be variously transformed and their placement on the time-line can also be quite arbitrary. Thus, a purely abstract rhythmic canon can be identified with an *ordered* hypergraph on the set of vertices $\{1, 2, \ldots, n\}$ whose edges correspond to the transformed copies of the leading motif. If the edges partition the set of vertices (a hypergraph is a *perfect matching*), then it is convenient to represent the canon by a *word* with same letters occupying positions of a fixed copy of the motif. If all copies are of the same size (a hypergraph is uniform), then each letter in the word occurs the same number of times. Words with this property are called *Gauss words*, in honor of the researcher who first used them in studying self-crossing curves on the plane.

There are many exciting problems about rhythmic canons. I will present a few of them during the talk. To get a foretaste, consider the following puzzle invented by Tom Johnson, a composer. Take a look at the word

$$ABCDCBCADBEEEDA.$$

It is an example of a *perfect rhythmic canon* $K(5, 3)$, that is, a *tiling* of the interval into five 3-term arithmetic progressions, each with a distinct gap. There are no such canons with two, three, four or six progressions, but it is known that $K(n, 3)$ exist for all $7 \leqslant n \leqslant 19$. In particular, there are 9257051746 different canons $K(19, 3)$. Is it true that for every $n \geqslant 7$ there is at least one perfect rhythmic canon $K(n, 3)$? Perfect canons $K(n, 4)$, built of 4-term arithmetic progressions of pairwise different gaps, are known to exist for all $15 \leqslant n \leqslant 23$. In particular, there are 19490 different canons $K(23, 4)$. What about canons $K(n, r)$ with $r \geqslant 5$? Do they exist for every fixed r and arbitrarily large n?

On b-acyclic chromatic number of cubic and subcubic graphs

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Let G be a graph. An acyclic k-coloring of G is a map $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$ and the subgraph induced by the vertices of any two colors $i, j \in \{1, \ldots, k\}$ is a forest. If every vertex v of a color class V_i misses a color $\ell_v \in \{1, \ldots, k\}$ in its closed neighborhood, then every $v \in V_i$ can be recolored with ℓ_v and we obtain a $(k-1)$-coloring of G. If a new coloring c' is also acyclic, then such a recoloring is an acyclic recoloring step and c' is in relation $\triangleleft_a$ with c. The acyclic b-chromatic number $A_b(G)$ of G is the maximum number of colors in an acyclic coloring where no acyclic recoloring step is possible. Equivalently, it is the maximum number of colors in a minimum element of the transitive closure of $\triangleleft_a$. In this talk, we develop the results presented in [1] by considering $A_b(G)$ of cubic and subcubic graphs.

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First-Fit Coloring of Forests in Random Arrival Model

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We study the performance of the First-Fit coloring algorithm on forests in the random arrival model. While this algorithm is known to use $\Theta(\log n)$ colors in the worst-case (adversarial) online model, its average-case performance under a random vertex permutation has been less understood.

We close this gap by providing tight asymptotic bounds. We show that for any forest with n vertices, the expected number of colors used by First-Fit is at most $(1 + o(1))\frac{\ln n}{\ln \ln n}$. Furthermore, we prove this bound is optimal by constructing a family of forests that requires $(1 - o(1))\frac{\ln n}{\ln \ln n}$ colors in expectation. Our result precisely characterizes the performance of First-Fit for this graph class, showing a modest but significant gain over the adversarial setting.

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On the distinguishing chromatic number in hereditary graph classes

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The distinguishing chromatic number of a graph G, denoted by $\chi_D(G)$, is the minimum number of colours in a proper vertex colouring of G that is preserved by the identity automorphism only. Collins and Trenk proved $\chi_D(G) \leq 2\Delta(G)$ for any connected graph G, and that equality holds for complete balanced bipartite graphs $K_{p,p}$ and for C_6.

In this talk, we show that the upper bound on $\chi_D(G)$ can be substantially reduced if we forbid some small graphs as induced subgraphs of G, that is, we study the distinguishing chromatic number in some hereditary graph classes.

Packing List-Colorings and the Proper Connection Number of Connected Graphs

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The *proper connection number* of a connected graph G is the minimum number of colors t required for a proper connected t-coloring of G; that is, an edge coloring of G such that between every pair of distinct vertices there exists a properly colored path.

We also consider list and list-packing versions of this number. Given a list L-edge-assignment of G, with $|L| = k$, an *L-packing proper connected coloring* of G is a collection of k mutually disjoint proper connected colorings $c_1, c_2, \ldots, c_k$ of the edges of G; that is, these colorings satisfy the conditions that for every vertex $v \in V(G)$ we have $c_i(v) \in L(v)$, and $c_i(v) \neq c_j(v)$ whenever $i \neq j$.

We discuss the origin of list-packing colorings and provide new results on this topic.

Interval colouring of oriented graphs

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An oriented graph is *interval colourable* if it admits an arc colouring with integers such that, for every vertex, the integers assigned to the in-arcs incident to this vertex are pairwise distinct, the integers assigned to the out-arcs incident to this vertex are also pairwise distinct, and both of these sets form intervals of integers. Since there exist oriented graphs that are not interval colourable, we analise the *interval colouring reorientation number* of an oriented graph D, denoted by $icr(D)$, defined as the minimum number of arcs of D that should be reversed so that a resulting oriented graph is interval colourable.

In this talk, we present properties and constraints of the interval colouring reorientation number, as well as its connections to other well-known parameters studied in the theory of graphs and digraphs.

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Totally Locally Irregular Decompositions of Graphs

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A locally irregular graph is a graph in which all adjacent vertices have distinct degrees. In article [1], the authors described the minimum number of locally irregular subgraphs into which a graph can be decomposed. This can be viewed as a graph coloring, where each color corresponds to a locally irregular subgraph. In [1], a total version of this problem is also defined.

In the problem of totally locally irregular decomposition of graphs, we aim to find the minimum number of colors in a total coloring of the graph such that, within each color class, all adjacent vertices have distinct total degrees.

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Packing coloring of graphs with long paths

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A *packing coloring* of a graph G is a mapping $c : V(G) \to \mathbb{N}$ such that any two distinct vertices assigned color i are at distance greater than i in G. This generalizes classical proper coloring by incorporating distance constraints that grow with the color index. The smallest integer k for which such a coloring exists using colors $1, \ldots, k$ is called the *packing chromatic number*, denoted $\chi_p(G)$.

We define a new class of graphs called *path-aligned graph products*, denoted by $P_n \lozenge_l G$. Let n and l be positive integers such that $l \mid n$, and let G be a connected vertex-transitive graph that contains a path P_l as a subgraph.The graph $P_n \lozenge_l G$ is constructed as follows.

- Start with the path P_n, with vertex sequence $v_1, v_2, \ldots, v_n$.
- Partition P_n into n/l consecutive, disjoint subpaths of length l, i.e., the i-th subpath is $P_l^{(i)} = (v_{(i-1)l+1}, \ldots, v_{il})$ for $i = 1, 2, \ldots, n/l$.
- For each i, take a copy $G^{(i)}$ of the graph G, and identify the subpath $P_l^{(i)} \subseteq P_n$ with a fixed copy of $P_l \subseteq G^{(i)}$. That is, the vertices of $P_l^{(i)}$ are merged with the corresponding vertices of the embedded path P_l in $G^{(i)}$.

We investigate the packing chromatic number χ_p of such constructions for various choices of G, including cycles and complete graphs, and determine exact values or bounds in these cases. Furthermore, we extend our results to selected classes of corona products, including generalized coronas, which share similar alignment properties.

Odd Coloring: Complexity and Algorithms

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An odd k-coloring of a graph $G = (V, E)$ is a proper k-coloring of G such that for every non-isolated vertex $v \in V$, there exists at least one color that appears an odd number of times in the open neighborhood of v. The minimum k for which G admits an odd k-coloring is called the odd chromatic number of G and is denoted by $\chi_o(G)$. Given a graph G and a positive integer k, Decide Odd Coloring Problem is to decide whether G admits an odd k-coloring. Decide Odd Coloring Problem is known to be NP-complete for general graphs [1]. In this paper, we strengthen this hardness result by proving that Decide Odd Coloring Problem remains NP complete for dually chordal graphs. On the positive side, we prove that for any proper interval graph G, the odd chromatic number satisfies $\omega(G) \leq \chi_o(G) \leq \omega(G) + 1$. We further characterize the proper interval graphs for which $\chi_o(G) = \omega(G)$, and those for which $\chi_o(G) = \omega(G) + 1$. We present a linear-time algorithm to compute the odd chromatic number of block graphs. Finally, we prove that the odd chromatic number of an interval graph G is either $\omega(G)$ or $\omega(G)+1$. Further, we characterize the interval graphs having $\chi_o(G) = \omega(G)$ and $\chi_o(G) = \omega(G) + 1$.

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Computational and algebraic approaches to open XOR-magic graphs

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A graph $G = (V, E)$ with $|V| = 2^n$ is called *(open) XOR-magic graph*, if it is connected and there exists a bijective labeling $\ell : V \to (\mathbb{Z}_2)^n$ such that for each vertex $v \in V$, sum of labels over (open) closed neighborhood of v is equal to $\mathbf{0}$. This labeling is a special case of group distance magic labeling of graphs.

Batal posed the following open problem: does it exist any even regular XOR-magic or odd regular open XOR-magic graph? In this talk, we will present positive answers to these questions, as well as a generalization about the existence of such graphs of order 2^n for each $n \geq 4$. Furthermore, we will present obtained algebraic approach to non-existence of open XOR-magic labelings and its application to various classes of circulant graphs.

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Weak and strong local irregularity of digraphs

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Local Irregularity Conjecture states that every connected graph, except special cacti, can be decomposed into at most three *locally irregular graphs*, i.e., graphs in which adjacent vertices have different degrees [1, 2]. The notion of local irregularity was defined for digraphs in several different ways. At the beginning of this talk we present the already known concepts of local irregularity for digraphs with motivations, main conjectures and known results. Then we introduce the following new methods of defining a *locally irregular digraph*. The first one, *weak local irregularity*, is based on distinguishing adjacent vertices by *indegree-outdegree pairs*, and the second one, *strong local irregularity*, asks for *different balanced degrees* (i.e., difference between the outdegree and the indegree of a vertex) of adjacent vertices. For both of these irregularities, we define locally irregular decompositions and colorings of digraphs. We also provide related conjectures on the minimum number of colors in weak and strong locally irregular colorings and support them with new results for various classes of digraphs.

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Majority Additive Coloring

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Majority additive coloring is a type of coloring where each vertex is assigned a number, and the sum of its neighbors' numbers, called the neighbor sum, is then computed. For the coloring to be valid, in the neighborhood of each vertex, at most half of its neighbors can share the same neighbor sum. Therefore, majority additive coloring is a combination of two known problems: additive coloring and majority coloring. The majority additive chromatic number, denoted by $\chi_{mac}(G)$, is the smallest number of colors required to achieve a majority additive coloring of G. We present several results regarding χ_{mac} for different types of graphs. For complete graphs and cycles, we have determined the exact value of the parameter, while for trees, we have found a tight upper bound. The main result of this work shows that for graphs with girth greater than 5, a sufficiently large maximum degree, and a minimum degree close to the maximum degree, it is sufficient to use only the numbers 1 and 2.

Odd coloring of k-trees

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For a graph G, an odd coloring of G is a proper coloring φ such that every non-isolated vertex v has a color c such that $|\varphi^{-1}(c) \cap N(v)|$ is an odd integer. A graph is said to be odd k-colorable if it admits an odd coloring with at most k colors. This notion was introduced by Petruševski and Škrekovski [2] in 2022, where they investigated odd coloring of planar graphs.

In this talk, we focus on odd coloring of k-trees. For a positive integer k, a graph which is obtained from K_{k+1} by recursively adding a vertex which is joined to a clique of order k is called a k-tree. For any $k \geq 1$, it is easy to see that there are infinitely many k-trees that are not odd $(k+1)$-colorable. On the other hand, according to a result by Cranston et al. [1], it follows that every graph of tree-width at most k is odd $(2k+1)$-colorable, and hence every k-tree is odd $(2k+1)$-colorable. We improve this bound by showing that every k-tree is odd $(k + 2\lfloor \log_2 k \rfloor + 3)$-colorable. Furthermore, when $k = 2, 3$, we show that every 2-tree is odd 4-colorable and that every 3-tree is odd 5-colorable, both of which are tight bounds. In particular, since every maximal outerplanar graph is a 2-tree, this implies that every maximal outerplanar graph is odd 4-colorable.

References

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Line graph orientations and list edge colorings of regular graphs

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The List Edge Coloring Conjecture states that for any graph G, the list chromatic index $ch'(G)$ equals the chromatic index $\chi'(G)$. A major breakthrough toward resolving this conjecture was Galvin's proof that it holds for bipartite graphs. It is natural to consider extending his coloring procedure to general graphs by decomposing them into bipartite subgraphs. However, such decompositions turn out to be incompatible with the method.

In 1996, Kahn proved that the conjecture holds asymptotically, establishing an upper bound of $\chi'(G) + o(\chi'(G))$. A later refinement by Häggkvist and Janssen, yielding a bound of $\chi'(G)+\widetilde{O}(\chi'(G)^{2/3})$, relies on the Alon–Tarsi polynomial method. This approach derives bounds from the existence of specific orientations of the line graph. Interestingly, such orientations can be constructed from those of bipartite subgraphs arising from natural decompositions. Therefore, any improvement of the result for bipartite graphs could potentially enhance the general bounds. Unfortunately, no such bipartite-specific improvements are currently known.

In this work, we explore this approach and present partial results on coloring line graphs of complete multipartite graphs.

List distinguishing index of graphs

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An edge colouring of a graph is called distinguishing if there is no non-trivial automorphism which preserves it. Distinguishing colourings gained quite a lot of attention since 1990s, and are still extensively studied. The most notable recent result in this area is the confirmation by Babai of the Infinite Motion Conjecture proposed by Tucker.

The talk will be about the list variant of this problem. We will present a general bound of $\Delta(G)-1$ for all connected graphs apart from some classified exceptions. This bound is optimal and it matches the best known bound for non-list colourings.

Then, we will discuss an improvement of the result of Lehner, Pilśniak, and Stawiski, which states that there is a distinguishing 3-edge-colouring of any connected regular graph except K_2. We prove that every at most countable, finite or infinite, connected regular graph of order at least 7 admits a distinguishing edge colouring from any set of lists of length 2.

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On the Orbital Chromatic Polynomial

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The orbital chromatic polynomial, introduced by Cameron and Kayibi in 2007, counts the number of proper λ-colorings of a graph modulo a group of symmetries. The polynomial has been investigated for specific graphs, including the Petersen graph, complete graphs, null graphs, paths, and cycles of small length. So far, no general formula for the orbital chromatic polynomial of the n-cycle for arbitrary n has been established.

In this talk we present such formula for the group of rotations and the full automorphism group of the n-cycle. As a side result, we obtain a new proof of Fermat's little theorem.

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Colouring cubic multipoles

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To study 3-edge-uncolourability of a cubic graph one can take a cut containing k edges and split the graph into two graph parts, called cubic k-poles. Each 3-edge-colouring of a k-pole induces a k-tuple of colours on the dangling edges, called boundary colouring. All colourings of the k-pole induce a (multi)set of boundary colourings, called colouring set. A colouring set contains only colourings satisfying parity lemma and the set has to be closed under Kempe switches. For $k \leq 5$ these two conditions are not only necessary but also sufficient. We will focus on the case where $k = 6$. We introduce a new equivalence relation that greatly reduces the number of colouring sets one needs to consider. We present the results of computational experiments using this equivalence relation.

For planar graphs Four Colour Theorem can be used to restrict colouring sets of k-poles. We show that certain generalised flow polynomials are an efficient tool to capture the number of colourings with given boundary. We explore conditions that Four Colour Theorem imposes on the polynomial and study k-poles that are close to "refuting" the Four Colour Theorem with respect to their polynomial coefficients.

Sufficient forbidden immersion conditions for graphs to be 7-colorable

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A graph H is an *immersion* of a graph G if there exist an injective function $f_1 : V(H) \to V(G)$ and a mappign f_2 from the edges of H to paths of G satisfying that

- for $uv \in E(H)$, $f_2(uv)$ is a path connecting $f_1(u)$ and $f_1(v)$ and
- for edges $e, e' \in E(H)$ with $e \neq e'$, $f_2(e)$ and $f_2(e')$ are pairwise edge-disjoint.

In analogy with Hadwiger's conjecture, Abu-Khzam and Langston [1] proposed the follwoing conjecture : every graph with no K_t as an immersion is $(t-1)$-colorable. Lescure and Meyniel [2] proved the conjecture for $t = 5, 6$ and DeVos, Kawarabayashi, Mohar, and Okamura [3] proved the conjecture for $t = 7$. In this talk, we discuss the conjecture for $t = 8$.

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Erdős-Pósa property of cycles that are far apart

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We prove that there exist functions f and g such that for all nonnegative integers k and d, for every graph G, either G contains k cycles such that vertices of different cycles have distance greater than d in G, or there exists a subset X of vertices of G, with $|X| \leqslant f(k)$ such that $G - B_G(X, g(d))$ is a forest, where $B_G(X, r)$ denotes the set of vertices of G having distance at most r from a vertex of X.

Edge-uncoverability by four perfect matchings in cubic graphs

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Several longstanding conjectures in graph theory, including the Cycle Double Cover Conjecture, can be reduced to the case of cubic graphs. A notable parameter in this context is the perfect matching index of a cubic graph, defined as the minimum number of perfect matchings needed to cover its edges. In particular, if these conjectures hold for cubic graphs with perfect matching index at least 5, they hold in general.

In this talk, we introduce several invariants that capture how far a cubic graph G is from being coverable by four perfect matchings. One such invariant is the *four perfect matching defect* of G, denoted by $d_{PM}(G)$, defined as the minimum number of edges of G not covered by four perfect matchings. Another is the *four matching cover defect* of G, denoted by $d_M(G)$, defined as the minimum sum of defects of matchings over all matching covers of G containing four matchings, where the *defect of a matching* is half of the number of vertices it leaves uncovered. We prove that $d_M(G) \leq d_{PM}(G)$. Furthermore, we show that for each integer k, there exists a cubic graph with $d_M(G) = d_{PM}(G) = k$; that is, there are cubic graphs that are far from being coverable by four perfect matchings. We also present another family of cubic graphs, in which, for each integer k, there exists a cubic graph with $2d_M(G) \leq d_{PM}(G) = 2k$.

Beyond offering new perspectives on the structure of cubic graphs, we also demonstrate that such measures yield partial results toward resolving these famous conjectures.

List extensions of majority edge colourings

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A *majority edge colouring* of a graph G is a colouring of the edges of G such that for each vertex v of G, at most half the edges incident with v have the same colour. More generally, for a natural number $k \geq 2$, a $1/k$*-majority edge-colouring* of a graph is a colouring of the edges of G such that for every colour c and every vertex v of G at most $1/k$ of the edges incident with v have the colour c. This notion was introduced in 2023 by Bock, Kalinowski, Pardey, Pilśniak, Rautenbach and Woźniak [1].

We investigate possible list extensions of generalised majority edge colourings. In particular, given a graph G, a list assignment L and a *majority tolerance* $\alpha \in (0, 1)$, an α*-majority* L*-colouring* of G is a colouring $\omega : E \to C$ from the given lists such that for every $v \in V$ and each $c \in C$, the number of edges coloured c which are incident with v does not exceed $\alpha \cdot d(v)$. We discuss some restrictions necessary to extend this notion to a more general setting with diversified $\alpha = \alpha(c)$ majority tolerances for distinct colours $c \in C$. In particular, for any list assignment $L : E \to 2^C$ with $\sum_{c \in L(e)} \alpha(c) \geq 1 + \varepsilon$ and $|L(e)| \leq \ell$ for each edge e, we show that there exists an α-majority L-colouring of G, provided that $\delta(G) = \Omega(\ell^2 \varepsilon^{-2})$.

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Soft Happy Colouring

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For a coloured graph G and $0 \leq \rho \leq 1$, a vertex v is ρ-happy if at least $\rho \deg(v)$ of its neighbours share its colour. The soft happy colouring problem seeks a colouring σ that extends a given precolouring and maximises the number of ρ-happy vertices [3]. This NP-hard problem is closely linked to community detection in graphs. For example, for a graph in the stochastic block model (SBM) and for suitable ρ, with high probability, complete soft happy colourings can be achieved by the planted community structure [1]. Moreover, for $0 \leq \rho_1 < \rho_2 \leq 1$, complete ρ_2-happy colourings achieve higher detection accuracy than complete ρ_1-happy colourings, and when ρ surpasses a critical threshold, it is unlikely to find a complete ρ-happy colouring with near-equal class sizes [2]. Finally, we survey existing algorithms and propose novel heuristic, local search, evolutionary, metaheuristic, and matheuristic approaches that enhance solution quality for soft happy colouring.

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List strong edge-colouring

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A strong edge-coloring of a graph is an edge-coloring in which every color class is an induced matching. The least number of colors needed for a strong edge-coloring of a graph is the *strong chromatic index*.

We consider the list version of the coloring and prove that the list strong chromatic index of graphs with maximum degree 3 is at most 10. This bound is tight and improves the previous bound of 11 colors ([1]).

Next, we consider graphs with maximum degree 4, where the best known bound for the list strong edge-coloring is 22 ([2]). We improve this result and establish an upper bound of 21 for the strong list chromatic index of subquartic graphs. Since there exist subquartic graphs whose strong chromatic index is 20, our bound is only one above the best possible.

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Irreducibility in distinguishing colourings

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We investigate the role of the Axiom of Choice and its weaker forms in distinguishing and proper colourings. In particular, we formulate conditions equivalent to AC in terms of such colourings in both vertex and edge variants. Moreover, we study the notion of *irreducible* distinguishing colourings, i.e. distinguishing colourings such that no two non-empty classes of colours may be merged to obtain another distinguishing colourings. One may view such as distinguishing colourings for which no colour is abundant.

Normal edge coloring

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A normal edge coloring of a cubic graph is a proper edge coloring, in which every edge is adjacent to edges colored with four distinct colors or to edges colored with two distinct colors. It is conjectured that 5 colors suffices for a normal edge coloring of any bridgeless cubic graph and this statement is equivalent to the Petersen Coloring Conjecture. Currently, we only know that any cubic graph admits a normal edge coloring with at most 7 colors.

We present new results regarding the normal coloring of special graph classes. In the second part, we introduce the study of the list version of the normal edge coloring. It turns out to be more restrictive and consequently more colors are needed. In particular, we show that there are cubic graphs which need at least 9 colors for a list normal edge coloring and there are bridgeless cubic graphs which need at least 8 colors.

Quasi-majority neighbor sum distinguishing edge-colorings

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An edge-coloring c of a graph G defines in a natural way a vertex-coloring $\sigma_c : V(G) \to \mathbb{N}$ by $\sigma_c(v) = \sum_{u \in N_G(v)} c(vu)$ for each $v \in V(G)$. The edge-coloring c is called neighbor sum distinguishing if $\sigma_c(u) \neq \sigma_c(v)$ for every $uv \in E(G)$. This type of edge-coloring is related to the 1-2-3 Conjecture, proved by Keusch [1].

We study neighbor sum distinguishing edge-coloring under additional local constraint, requiring the edge-coloring to be quasi-majority. A k-edge-coloring of G is called quasi-majority if for every $v \in V(G)$ and every $\alpha \in [k]$, at most $\left\lceil \frac{d(v)}{2} \right\rceil$ edges incident to v are colored with α.

A k-edge-coloring of G is called quasi-majority neighbor sum distinguishing if it is quasi-majority and neighbor sum distinguishing. The smallest k for which G admits such a coloring is denoted by $\chi_{\sum}^{QM}(G)$. A graph is nice if it has no component isomorphic to K_2. We show that $\chi_{\sum}^{QM}(G) \leq 12$ for every nice G. This bound improves to 6 for nice bipartite graphs and to 7 for nice graphs of maximum degree at most four. Moreover, we determine the exact value of $\chi_{\sum}^{QM}(G)$ for complete graphs, complete bipartite graphs, and trees.

References

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Odd independent sets and strong odd colorings of graphs

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We say that an $S \subset V(G)$ is an odd independent set in graph G if it is independent (induces no edges) and every vertex in $V \setminus S$ is adjacent either to no vertex of S or to an odd number of vertices of S. The largest cardinality of such a set is termed the odd independence number of G.

A strong odd coloring of G is a partition of the vertex set into odd independent sets; the corresponding parameter (minimum number of colors) is called strong odd chromatic number.

Beside many results concerning these notions, we also offer a large number of open problems for future research.

On high-girth high-chromatic subgraphs of Burling graphs

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A well-known conjecture of Erdős and Hajnal asserts that for any g and k, there is a finite number $f(g,k)$ such that every graph with chromatic number at least $f(g,k)$ contains a subgraph with girth at least g and chromatic number at least k. Rödl (1977) proved the conjecture for $g = 4$, in particular showing that $f(4,k)$ is bounded from above by a tower of ks of height $O(k^2 \log k)$. The conjecture remains open for $g \geq 5$.

A construction of triangle-free high-chromatic graphs due to Burling (1965) was used in the last decade to provide counterexamples to several conjectures and was shown to have various unexpected properties. We show that while Burling graphs do satisfy the aforesaid Erdős–Hajnal conjecture, they provide the first non-trivial lower bound on the growth of $f(g,k)$. Specifically, if $T(k)$ denotes the tower of 2s of height k, we prove that the Burling graph with chromatic number $T(k - O(1))$ has no subgraph with girth 5 and chromatic number at least k, showing that $f(5,k) > T(k - O(1))$.

A key tool for the proof is a combinatorial game in which two players, Builder and Chooser, alternate turns to build a graph vertex by vertex as follows: Builder introduces a new vertex with edges to all previous vertices and then partitions the entire edge set into two subsets, after which Chooser deletes one of the two subsets. Builder attempts to build a clique of size k, while Chooser attempts to prevent that. We prove upper and lower bounds of the form $T(k \pm O(1))$ on how many turns Builder needs to guarantee a k-clique.