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Extremal Graph Theory

Book of abstracts

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List of talks

Outdegree conditions forcing directed cycles

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In 2010, Kelly, Kühn and Osthus [2] made a conjecture on the minimum semidegree which forces an oriented graph to contain a directed cycle of a given length at least 4. The conjecture was proven by its authors for cycles of length not divisible by 3 and in [1] for other cycles. We consider an analogous problem, but without the assumption on the minimum indegree, and prove an optimal bound on the minimum outdegree which forces an oriented graph to contain a directed cycle of a given large enough length.

References

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[2] L. Kelly, D. Kühn, D. Osthus, Cycles of given length in oriented graphs, *J. Combin. Theory Ser. B* 100 (2010), 251–264.

Largest planar graphs of diameter 3 and fixed maximum degree

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The degree diameter problem asks for the maximum possible number of vertices in a graph of maximum degree Δ and diameter D. In this paper, we focus on planar graphs of diameter 3. Fellows, Hell and Seyffarth [1] proved that for all $\Delta \geq 8$, the maximum number $\mathrm{np}_{\Delta,D}$ of vertices of a planar graph with maximum degree at most Δ and diameter at most 3 satisfies $\frac{9}{2}\Delta - 3 \leq \mathrm{np}_{\Delta,3} \leq 8\Delta + 12$. We show that the given lower bound is tight up to an additive constant, by proving that there exists a constant $c > 0$ such that $\mathrm{np}_{\Delta,3} \leq \frac{9}{2}\Delta + c$ for every $\Delta \geq 0$. Our proof consists in a reduction to the fractional maximum matching problem on a specific class of planar graphs, for which we show that the optimal solution is $\frac{9}{2}$, and characterize all graphs attaining this bound.

References

[1] Michael R. Fellows, Pavol Hell, Karen Seyffarth Large planar graphs with given diameter and maximum degree, *Discrete Applied Mathematics* 1995 61(2) pp.133-153.

Extremal problems on planar graphs

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Let $\mathrm{ex}_{\mathcal{P}}(n, T, H)$ denote the maximum number of copies of T in an n-vertex planar graph which does not contain H as a subgraph. When $T = K_2$, $\mathrm{ex}_{\mathcal{P}}(n, T, H)$ is the planar Turán number of H, denoted by $\mathrm{ex}_{\mathcal{P}}(n, H)$. The topic of extremal planar graphs was initiated by Dowden (2016) [1]. He obtained sharp upper bound for both $\mathrm{ex}_{\mathcal{P}}(n, C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5)$. In [2], we gave a sharp upper bound $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7$, for all $n \geq 18$. We also pose a conjecture on $\mathrm{ex}_{\mathcal{P}}(n, C_k)$, for $k \geq 7$.

We [3] proved that for every integer $n \geq 6$, $\mathrm{ex}_{\mathcal{P}}(n, C_5, \emptyset)$ is $2n^2 - 10n + 12 + \mathbb{1}_{n=7}$.

And (see [4]) for every fixed $k \geq 3$, $\mathrm{ex}_{\mathcal{P}}(n, C_{2k}, \emptyset)$ is $n^k/k^k + o(n^k)$. In this lecture, we present more recent similar results related to cycles and paths.

References

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On the Turán number of the expansion of the t-fan

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The t-fan, F_t, is the graph on $2t + 1$ vertices consisting of t triangles that intersect at exactly one common vertex. For a given graph F, the r-expansion F^r of F is the r-uniform hypergraph obtained from F by adding $r - 2$ distinct new vertices to each edge of F. The Turán number of an r-uniform hypergraph $\mathcal{H}$, $ex_r(n, \mathcal{H})$, is the maximum number of hyperedges an r-uniform n-vertex hypergraph can have without containing $\mathcal{H}$ as a subhypergraph. We determine the Turán number of the 3-expansion of the t-fan for sufficiently large n. Namely, we show that $\mathrm{ex}_3(n, \mathcal{F}_t^3) = \binom{n}{3} - \binom{n-t}{3}$.

Unavoidable subgraphs in digraphs with large out-degrees

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We ask the question, which oriented trees T must be contained as subgraphs in every finite directed graph of sufficiently large minimum out-degree. We formulate the following simple condition: all vertices in T of in-degree at least 2 must be on the same 'level' in the natural height function of T. We prove this condition to be necessary and conjecture it to be sufficient. In support of our conjecture, we prove it for a fairly general class of trees.

An essential tool in the latter proof, and a question interesting in its own right, is finding large subdivided in-stars in a directed graph of large minimum out-degree. We conjecture that any digraph and oriented graph of minimum out-degree at least $k\ell$ and $k\ell/2$, respectively, contains the $(k-1)$-subdivision of the in-star with ℓ leaves as a subgraph; this would be tight and generalizes a conjecture of Thomassé. We prove this for digraphs and $k = 2$ up to a factor of less than 4.

Random embeddings of bounded degree trees with optimal spread

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A seminal result of Komlós, Sárközy, and Szemerédi [1] states that any n-vertex graph G with minimum degree at least $(1/2+\alpha)n$ contains every n-vertex tree T of bounded degree. Recently, Pham, Sah, Sawhney, and Simkin [2] extended this result to show that such graphs G in fact support an optimally spread distribution on copies of a given T, which implies, using the recent breakthroughs on the Kahn-Kalai conjecture, the robustness result that T is a subgraph of sparse random subgraphs of G as well. Pham, Sah, Sawhney, and Simkin construct their optimally spread distribution by following closely the original proof of the Komlós-Sárközy-Szemerédi theorem which uses the blow-up lemma and the Szemerédi regularity lemma. We give an alternative, regularity-free construction that instead uses the Komlós-Sárközy-Szemerédi theorem (which has a regularity-free proof due to Kathapurkar and Montgomery) as a black-box. Our proof is based on the simple and general insight that, if G has linear minimum degree, almost all constant sized subgraphs of G inherit the same minimum degree condition that G has.

References

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Constructions of Turán systems that are tight up to a multiplicative constant

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The Turán density $t(s,r)$ is the asymptotically smallest edge density of an r-graph for which every set of s vertices contains at least one edge. The question of estimating this function received a lot of attention since it was first raised by Turán in 1941. A trivial lower bound is $t(s,r) \geq 1/\binom{s}{s-r}$. In the early 1990s, de Caen [1] conjectured that $t(r+1,r)$ grows faster than $O(1/r)$ and offered 500 Canadian dollars for resolving this question.

I will give an overview of this area and present a construction from [2] disproving this conjecture by showing more strongly that for every integer R there is C such that $t(r+R,r) \leq C/\binom{r+R}{R}$, that is, the trivial lower bound is tight for every fixed R up to a multiplicative constant $C = C(R)$.

References

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Distinct degrees and homogeneous sets in graphs

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In this talk we investigate the extremal relationship between the order of a largest homogeneous set (clique or independent set) in a graph G and the maximal number of distinct degrees that appear in an induced subgraph of G, denoted by $\hom(G)$ and $f(G)$ respectively. This topic has been well studied by several researchers over the last 40 years, beginning with Erdős, Faudree and Sós in the regime when $\hom(G) = O\big(\log|G|\big)$.

Our main theorem asymptotically settles this question, improving on multiple earlier estimates. More precisely, we show that any n-vertex graph G satisfies:

$$f(G) \geq \min\left(\sqrt[3]{\frac{n^2}{\hom(G)}}\ ,\ \frac{n}{\hom(G)}\right) \cdot n^{-o(1)}.$$

This relationship is tight (up to the $n^{o(1)}$ term) for all possible values of $\hom(G)$, as demonstrated by appropriately generated Erdős−Renyi random graphs. Our approach to lower bounding $f(G)$ proceeds via a translation into an (almost) equivalent probabilistic problem, which is effective for arbitrary graphs.

Cycle lengths in graphs of given minimum degree

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We present minimum degree conditions which forces a graph to contain a cycle of length ℓ modulo k for fixed k and ℓ. Our outcomes improve the results of Gao, Huo, Liu and Ma [1]. Consequently, we determine the maximum number of edges in a graph that does not contain a cycle of length 0 modulo k for odd k.

References

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Antidirected paths in oriented graphs

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We show that for any integer $k \geq 4$, every oriented graph with minimum semidegree bigger than $\frac{1}{2}(k-1+\sqrt{k-3})$ contains an antidirected path of length k. Consequently, every oriented graph on n vertices with more than $(k-1+\sqrt{k-3})n$ edges contains an antidirected path of length k. This asymptotically proves the antidirected path version of a conjecture of Stein and of a conjecture of Addario-Berry, Havet, Linhares Sales, Reed and Thomassé, respectively.

References

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