On high-girth high-chromatic subgraphs of Burling graphs

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A well-known conjecture of Erdős and Hajnal asserts that for any g and k, there is a finite number $f(g,k)$ such that every graph with chromatic number at least $f(g,k)$ contains a subgraph with girth at least g and chromatic number at least k. Rödl (1977) proved the conjecture for $g = 4$, in particular showing that $f(4,k)$ is bounded from above by a tower of ks of height $O(k^2 \log k)$. The conjecture remains open for $g \geq 5$.

A construction of triangle-free high-chromatic graphs due to Burling (1965) was used in the last decade to provide counterexamples to several conjectures and was shown to have various unexpected properties. We show that while Burling graphs do satisfy the aforesaid Erdős–Hajnal conjecture, they provide the first non-trivial lower bound on the growth of $f(g,k)$. Specifically, if $T(k)$ denotes the tower of 2s of height k, we prove that the Burling graph with chromatic number $T(k - O(1))$ has no subgraph with girth 5 and chromatic number at least k, showing that $f(5,k) > T(k - O(1))$.

A key tool for the proof is a combinatorial game in which two players, Builder and Chooser, alternate turns to build a graph vertex by vertex as follows: Builder introduces a new vertex with edges to all previous vertices and then partitions the entire edge set into two subsets, after which Chooser deletes one of the two subsets. Builder attempts to build a clique of size k, while Chooser attempts to prevent that. We prove upper and lower bounds of the form $T(k \pm O(1))$ on how many turns Builder needs to guarantee a k-clique.