Distinct degrees and homogeneous sets in graphs

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In this talk we investigate the extremal relationship between the order of a largest homogeneous set (clique or independent set) in a graph G and the maximal number of distinct degrees that appear in an induced subgraph of G, denoted by $\hom(G)$ and $f(G)$ respectively. This topic has been well studied by several researchers over the last 40 years, beginning with Erdős, Faudree and Sós in the regime when $\hom(G) = O\big(\log|G|\big)$.

Our main theorem asymptotically settles this question, improving on multiple earlier estimates. More precisely, we show that any n-vertex graph G satisfies:

$$f(G) \geq \min\left(\sqrt[3]{\frac{n^2}{\hom(G)}}\ ,\ \frac{n}{\hom(G)}\right) \cdot n^{-o(1)}.$$

This relationship is tight (up to the $n^{o(1)}$ term) for all possible values of $\hom(G)$, as demonstrated by appropriately generated Erdős–Renyi random graphs. Our approach to lower bounding $f(G)$ proceeds via a translation into an (almost) equivalent probabilistic problem, which is effective for arbitrary graphs.