Gauss words and rhythmic canons

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A *rhythmic canon* is a combinatorial structure consisting of repeating copies of the same *motif.* These copies may be variously transformed and their placement on the time-line can also be quite arbitrary. Thus, a purely abstract rhythmic canon can be identified with an *ordered* hypergraph on the set of vertices $\{1, 2, \ldots, n\}$ whose edges correspond to the transformed copies of the leading motif. If the edges partition the set of vertices (a hypergraph is a *perfect matching*), then it is convenient to represent the canon by a *word* with same letters occupying positions of a fixed copy of the motif. If all copies are of the same size (a hypergraph is uniform), then each letter in the word occurs the same number of times. Words with this property are called *Gauss words*, in honor of the researcher who first used them in studying self-crossing curves on the plane.

There are many exciting problems about rhythmic canons. I will present a few of them during the talk. To get a foretaste, consider the following puzzle invented by Tom Johnson, a composer. Take a look at the word

$$ABCDCBCADBEEEDA.$$

It is an example of a *perfect rhythmic canon* $K(5, 3)$, that is, a *tiling* of the interval into five 3-term arithmetic progressions, each with a distinct gap. There are no such canons with two, three, four or six progressions, but it is known that $K(n, 3)$ exist for all $7 \leqslant n \leqslant 19$. In particular, there are 9257051746 different canons $K(19, 3)$. Is it true that for every $n \geqslant 7$ there is at least one perfect rhythmic canon $K(n, 3)$? Perfect canons $K(n, 4)$, built of 4-term arithmetic progressions of pairwise different gaps, are known to exist for all $15 \leqslant n \leqslant 23$. In particular, there are 19490 different canons $K(23, 4)$. What about canons $K(n, r)$ with $r \geqslant 5$? Do they exist for every fixed r and arbitrarily large n?